INTERPOLATING SEQUENCES IN POLYDISKS

BY

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ABSTRACT. Let D^n be the unit polydisk in $\mathbb{C}^n$, A be a uniform algebra, $H^\infty(D^n, A)$, the space of bounded analytic A valued functions on D^n, $l^\infty A$, the space of bounded sequences of elements in A. A sequence, $S = \{a_i\}_{i=1}^\infty \subset D^n$ will be called an interpolating sequence with respect to A if the map $T: H^\infty(D^n, A) \longrightarrow l^\infty A$ given by $T(f) = \{f(a_i)\}_{i=1}^\infty$ is surjective. In 1958, L. Carleson showed that for $n = 1, S$ is interpolating w.r.t. $\mathbb{C}$ iff S satisfies a certain zero-one interpolation property called uniform separation. We generalize this result to cases where $n > 1$ and $A \neq \mathbb{C}$. In particular, we show that if $S \subset D^n$ is uniformly separated and $S \subset W_1 \times W_2 \times \cdots \times W_n$ (where each W_j is a region in D lying between two circular arcs which intersect twice on the boundary of D) then S is an interpolating sequence w.r.t. any uniform algebra. If $S \subset D^n$ is uniformly separated and $S \subset D \times W_2 \times \cdots \times W_n$ then S is interpolating w.r.t. $\mathbb{C}$. Other examples and generalizations of interpolating sequences are discussed.

Let A be a uniform algebra, i.e. a closed subalgebra of the continuous functions on a compact Hausdorff space which separates points, contains constants, and is equipped with the sup-norm topology. (See [2] and [7] for detailed descriptions of uniform algebras.) Let D^n be the (open) unit polydisk in $\mathbb{C}^n$; let $H^\infty(D^n, A)$ be the Banach algebra of bounded, analytic, A-valued functions on D^n equipped with the sup-norm (see [5, pp. 224–232] for a description of such functions); and let $l^\infty A$ be the Banach algebra of bounded sequences of elements of A. If $S = \{a_i\}_{i=1}^\infty \subset D^n$ is a sequence of points in D^n, we can define a map, $T: H^\infty(D^n, A) \to l^\infty A$ by $T(f) = \{f(a_i)\}_{i=1}^\infty$. We will say S is an interpolating sequence w.r.t. A if T is surjective. If S is an interpolating sequence w.r.t. $\mathbb{C}$, we will simply say S is an interpolating sequence, and if S is an

interpolating sequence w.r.t. every uniform algebra, we still say S is a general interpolating sequence.

If $S = \{a_i\}_{i=1}^{\infty} \subset D^n$ is an interpolating sequence, then it is known that S must be uniformly separated, i.e. there exists a constant M and functions f_1, $f_2, \cdots \in H^{\infty}(D^n)$ such that for all i, $\|f_i\| \leqslant M$ and $f_i(a_i) = 1$ while f_i is zero on the remaining points of S. (We will use $H^{\infty}(D^n)$ for $H^{\infty}(D^n, \mathbf{C})$.) In 1958, L. Carleson [3] showed that for $S \subset D$ $(D = D^1)$, uniform separation is a necessary and sufficient condition that S be an interpolating sequence (w.r.t. $\mathbf{C}$). In this paper we attempt to extend Carleson's results to sequences in D^n and to general interpolating sequences. We succeed to the extent of showing that uniform separation is sufficient in a large class of sequences in D^n and D.

After introducing notation in §1, we review in §2 some of the concepts and results presented in [10]. §§3 and 4 are devoted to a number of techniques for constructing general interpolating sequences. §5 contains the main result of this paper demonstrating that uniformly separated sequences in "wedges" of D^n are general interpolating sequences. §6 presents some miscellaneous examples of general interpolating sequences, and the last section indicates some directions in which all these results might be generalized.

1. Throughout this paper, A will be a uniform algebra. We will let m_A denote the maximal ideal space or spectrum of A. m_A is of course a subset of A^*, the dual space of A, and as such can be equipped with the weak-* topology in which it is a compact Hausdorff space. We will identify elements of A with their Gelfand transforms, so we may consider elements of A as functions on m_A.

It should be noted that m_A is naturally supplied with two topologies as a subset of A^*—the weak-* topology and the norm topology. We will sometimes have occasion to use both, and it may also happen that subsets of m_A are naturally associated with other topologies. Throughout this paper, if $Q \subset m_A$, we will consistently use the symbol $\mathrm{Cl}(Q)$ to represent the weak-* closure of Q and will explicitly mention all other topologies. We will also use the symbol $\mathrm{Hk}(Q)$ for the hull of the kernel of Q, i.e.

$$\mathrm{Hk}(Q) = \{s \in m_A : x(s) = 0 \text{ for every } x \in A \text{ for which } x(Q) = 0\}.$$

The norm topology for m_A is equivalent to the one induced by the "pseudo-hyperbolic" metric:

$$\rho(z, w) = \mathrm{Sup}\, \{|x(z)| : x \in A, \|x\| \leqslant 1 \text{ and } x(w) = 0\}.$$

We will use D to represent the open unit disk in the complex plane, $\mathbf{C}$, and D^n for the open unit polydisk in n-dimensional complex space $\mathbf{C}^n$. T^n will refer to the essential boundary of D^n, i.e.

$$T^n = \{\langle z^{(1)}, z^{(2)}, \cdots, z^{(n)}\rangle \in \mathbf{C}^n \colon |z^{(j)}| = 1 \text{ for every } j, j = 1, \cdots, n\}.$$

$H^\infty(D)$ and $H^\infty(D^n)$ are, of course, the algebras of bounded analytic complex valued functions on D and D^n respectively. $H^\infty(D, A)$ and $H^\infty(D^n, A)$ will be the algebras of bounded analytic A-valued functions on D and D^n respectively.

m_1, m_n will be written instead of $m_{H^\infty(D)}$ and $m_{H^\infty(D^n)}$ and points in D (or D^n) will be identified with their associated point evaluation in m_1 (or m_n). Finally, we will often (but not always) use a subscript for the pseudo-hyperbolic metrics on m_n, e.g. ρ_1 will be the pseudo-hyperbolic metric on m_1. These metrics can be described for the cases where points are contained inside the polydisk involved, i.e. for $n = 1$, $z, w \in D$, then $\rho_1(z, w) = |(z - w)/(1 - z\bar{w})|$. For $n > 1$

$$\langle z^{(1)}, z^{(2)}, \cdots, z^{(n)}\rangle, \langle w^{(1)}, \cdots, w^{(n)}\rangle \in D^n;$$

then

$$\rho_n(\langle z^{(1)}, \cdots, z^{(n)}\rangle, \langle w^{(1)}, \cdots, w^{(n)}\rangle)$$

$$= \text{Max } \{\rho_1(z^{(j)}, w^{(j)})\colon j = 1, 2, \cdots, n\}.$$

In other words, the pseudo-hyperbolic distance between two points in D^n is the maximum of the pseudo-hyperbolic distances of the projections of the points on each of the coordinate axes.

2. **General interpolating, uniformly separated, and α-separated sequences.**
If S is an interpolating sequence w.r.t. A, the map $T\colon H^\infty(D^n, A) \to l^\infty A$ described above is surjective, and the open mapping theorem implies the existence of a constant M_A such that for any element $\{X_i\}_{i=1}^\infty \in l^\infty A$, there exists $f \in H^\infty(D^n, A)$ such that $\|f\| \leqslant M_A(\text{Sup}_i \|X_i\|)$ and $f(a_i) = X_i$, $i = 1, 2, \cdots$. Any such M_A will be called an interpolating bound for S w.r.t. A.

PROPOSITION 2.1. *If S is a general interpolating sequence, there exists a single constant M with the property that if A is a uniform algebra, there is an interpolating bound for S w.r.t. A which is less than M.*

PROOF. For any uniform algebra A, let $N_A = \text{Inf } \{M\colon M$ is an interpolating bound for S w.r.t. $A\}$. Suppose the theorem is false, then we can choose uniform algebras $A_1, A_2, \cdots$ so that $N_{A_k} \geqslant k$ for $k = 1, 2, \cdots$. Consider the Cartesian product $Q = A_1 \times A_2 \times A_3 \times \cdots$, and the algebra $A = \{x = \langle x_1, x_2, \cdots\rangle \in Q\colon \text{Sup}_i \|x_i\|_{A_i} < \infty\}$. A will be a Banach algebra under coordinate-wise addition and multiplication and the norm given by $\|x\| = \|\langle x_1, x_2, x_3, \cdots\rangle\| = \text{Sup}_i \{\|x_i\|_{A_i}\}$. Since A_i is a uniform algebra $\|x_i^2\|_{A_i} = (\|x_i\|_{A_i})^2$ for all $x_i \in A_i$, hence $\|x^2\| = \|x\|^2$ for all $x \in A$, and consequently

the Gelfand transform of A is an isometric isomorphism, so A is isomorphic to a uniform algebra. We note that the projection maps $\pi_j : A \longrightarrow A_j$ are all linear, of norm one. Let M be a general interpolating bound for S with respect to A. Fix $k > M$, and consider $y_1, y_2, \cdots \in A_k$ such that $\|y_i\|_{A_k} \leqslant 1$. Define $t_j \in A$ by $t_j = \langle 0, 0, \cdots, 0, y_j, 0, \cdots \rangle$ (where the y_j is in the kth place); then $\|t_j\| \leqslant 1$ for all j. Hence there is a bounded analytic function $f : D^n \longrightarrow A$ such that $f(a_j) = t_j$ for $j = 1, 2, \cdots$ and $\|f(z)\| \leqslant M$ for all $z \in D^n$. Then $g = \pi_k \circ f$ will be a bounded analytic function, $g : D^n \longrightarrow A_k$ such that $g(a_j) = y_j$ for $j = 1, 2, \cdots$, and $\|g(z)\|_{A_k} \leqslant M$ for all $z \in D^n$. Therefore $N_{A_k} \leqslant M$, which contradicts $k \leqslant N_{A_k}$ and $M < k$, so that the A_k's must not have existed in the first place.

DEFINITION 2.2. Any constant M with the property of Proposition 1.1 will be called a general interpolating bound for S.

DEFINITION 2.3. A sequence $S = \{a_i\}_{i=1}^\infty \subset D^n$ will be called a uniformly separated (u.s.) sequence if there exists a constant $M > 0$ and functions f_1, $f_2, \cdots \in H^\infty(D^n)$ with the property that $\|f_i\| \leqslant M$ for all i, and $f_i(a_i) = 1$ while $f_i(a_j) = 0$ for $i \neq j$, $i = 1, 2, \cdots$. If S is u.s., the constant M will be called a u.s. bound for S.

We have noted in the introduction that if $S \subset D$, then S is interpolating iff S is u.s. In this case, by means of Blaschke products, we see that S is a u.s. sequence in D iff

$$\operatorname*{Inf}_i \prod_{j=1; j \neq i}^\infty \left| \frac{a_i - a_j}{1 - a_i \bar{a}_j} \right| > 0.$$

This last condition is equivalent to the following property: There exist constants $M > 0$ and $p > 0$ and functions $\theta_1, \theta_2, \cdots \in H^\infty(D)$ such that
  (1) $\|\theta_i\| \leqslant 1$ for all i.
  (2) $\theta_i(a_i) = 0$ while $|\theta_i(a_j)| > p$ if $i \neq j$.
  (3) $\Sigma_{j=1}^\infty 1 - |\theta_j(a_i)| \leqslant M$ for all i.
  (4) $\Sigma_{j=1}^\infty |1 - \theta_j(z)|$ converges uniformly on compact subsets of D.
(The function $\theta_j(z)$ is just $\bar{a}_j/|a_j| \cdot (a_j - z)/(1 - \bar{a}_j z)$.) We define a similar, stronger, property which is sufficient for general interpolation.

DEFINITION 2.4. A sequence $S = \{a_i\}_{i=1}^\infty \subset D^n$ will be called α-separated if there exist constants $m, p > 0$ and functions $\alpha_1, \alpha_2, \alpha_3, \cdots \in H^\infty(D^n)$ such that
  (1) $\|\alpha_i\| \leqslant 1$ for all i.
  (2) $\alpha_i(a_i) = 0$ while $|\alpha_i(a_j)| > p$ for $i \neq j$.
  (3) $\Sigma_{j=1}^\infty |1 - \alpha_j(a_i)| \leqslant m$ for all i.
  (4) $\Sigma_{j=1}^\infty |1 - \alpha_j(z)|$ converges uniformly on compact subsets of D^n.

Note. Definition 2.4 includes sequences in polydisks, and the four conditions differ from the four listed above only in the placement of the absolute value signs in (3).

THEOREM 2.5. *If $S = \{a_i\}_{i=1}^{\infty} \subset D^n$ is α-separated, then S is a general interpolating sequence with bound depending only on the parameters m, p of the definition of α-separation.*

PROOF. This is Theorem 2.2 of [10].

3. **Jiggling theorems.** We would like to apply Theorem 2.5 to a fairly general class of sequences in D^n. In order to do this, we will have to develop some techniques for constructing new interpolating sequences from old ones. The first of these techniques involves "jiggling" the points of a known interpolating sequence. The other techniques include taking unions and Cartesian products of two or more sequences. The jiggling theorems provide a small neighborhood around each point of an interpolating sequence in which that point is free to move without affecting the interpolating property. They are based on the following two elementary lemmas.

LEMMA 3.1. *Suppose A, B are two Banach spaces and $T \in \mathrm{Hom}\,(A, B)$ and $E \subset B$ is the unit ball of B. Suppose there exist constants $0 < k < 1$ and $K > 0$ such that for all $Y \in E$, there is an $X \in A$ such that $\|TX - Y\| \leqslant k$ and $\|X\| \leqslant K$, then T is onto, and for all Y in B there exists X in A such that $TX = Y$ and $\|X\| \leqslant K/(1 - k)$.*

PROOF. See for example Theorem 1.2 of [1] for a generalization.

LEMMA 3.2. *If $f: D^n \longrightarrow A$ is analytic, $\|f(z)\| \leqslant 1$ for all $z \in D^n$, then for $z, w \in D^n$, $\|f(z) - f(w)\| \leqslant 2\rho(z, w)$.*

PROOF. Let $z, w \in D^n$ be fixed, and choose $\phi \in A^*$ (A^* is the dual space of A) such that $\|\phi\| = 1$ and $\phi(f(z) - f(w)) = \|f(z) - f(w)\|$. Let $F = \phi \circ f$, then $F \in H^{\infty}(D^n)$ and $\|F\| \leqslant 1$, so

$$\left| \frac{F(z) - F(w)}{1 - F(z)\overline{F(w)}} \right| \leqslant \rho(z, w),$$

hence $|F(z) - F(w)| \leqslant 2\rho(z, w)$. Consequently $\|f(z) - f(w)\| \leqslant 2\rho(z, w)$.

THEOREM 3.3. *If $S = \{a_i\}_{i=1}^{\infty} \subset D^n$ is an interpolating sequence w.r.t. A with interpolating bound, M_A(w.r.t. A), and $\epsilon \leqslant 1/4M_A$, and if we choose $b_i \in D^n$ such that $\rho_n(a_i, b_i) \leqslant \epsilon$ for all i, $i = 1, 2, \cdots$ then $T = \{b_i\}_{i=1}^{\infty}$ is an interpolating sequence with bound $2M_A$ (w.r.t. A).*

PROOF. Let E be the unit ball of $l^\infty A$. If $\{X_i\}_{i=1}^\infty \in E$ and $f: D^n \longrightarrow A$ is an analytic function with $f(a_i) = X_i$ and $\|f\| \leqslant M_A$; then

$$\|f(b_i) - X_i\| = \|f(b_i) - f(a_i)\| \leqslant 2M_A \rho_n(a_i, b_i) \leqslant \tfrac{1}{2}.$$

Hence Lemma 3.1 applies and T is an interpolating sequence w.r.t. A with bound $2M_A$.

COROLLARY 3.4 (JIGGLING THEOREM FOR GENERAL INTERPOLATING SEQUENCES). *If $S = \{a_i\}_{i=1}^\infty D^n$ is a general interpolating sequence with general interpolating bound M, and $\epsilon = 1/4M$, and if we choose $b_i \in D^n$ such that $\rho_n(a_i, b_i) \leqslant \epsilon$ for $i = 1, 2, \cdots$, then $T = \{b_i\}_{i=1}^\infty$ is a general interpolating sequence with bound $2M$.*

The jiggling theorems tell us that a sequence which is "close" to a given interpolating sequence is also interpolating. We can attempt an analogous characterization of u.s. sequences. We remark that although we generally take the terms "interpolating" and "uniformly separated" to apply to infinite sequences, they can also be applied to finite sets of points. Shapiro and Shields [14] showed that a u.s. sequence (finite or infinite) in D with u.s. bound $1/\delta$ is interpolating, with interpolating bound $(2/\delta^5)(1 - 2 \log \delta)$ or, put another way, if $S \subset D$ is a u.s. sequence with u.s. bound $1/\delta$ and h is a bounded function on S, $\|h\|_{\text{Sup}} \leqslant \delta^5/2(1 - 2 \log \delta)$, then there exists $f \in H^\infty(D)$ such that $f \restriction S = h$ and $\|f\| \leqslant 1$. We will use the last estimate frequently, so in order to simplify the notation we will write

$$\kappa(\delta) = \delta^5/2(1 - 2 \log \delta).$$

The following theorem shows that if a sequence in D is "almost" uniformly separated, then it is indeed u.s.

THEOREM 3.5. *Let $S = \{a_i\}_{i=1}^\infty \subset D$ and suppose there exists a constant m, $0 < m < 1$ and functions $f_1, f_2, \cdots \in H^\infty(D)$ such that*

$$\|f_i\| \leqslant 1 \quad \text{for } i = 1, 2, \cdots; \qquad |f_i(a_i)| > m \quad \text{for } i = 1, 2, \cdots;$$

and

$$|f_i(a_j)| \leqslant \epsilon_0 = (m/2)\kappa(m/2) \quad \text{for } i \neq j.$$

Then S is uniformly separated (hence interpolating) and S has u.s. bound $2/m$.

PROOF. For $N = 2, 3, \cdots$ we show that any set T of N distinct points of S is uniformly separated with u.s. bound $2/m$. The theorem then follows from standard pointwise limit arguments for $H^\infty(D)$.

We proceed by induction. If $N = 2$, let z, w be two distinct points in S. The hypothesis of the theorem guarantees the existence of $f \in H^\infty(D)$, $\|f\| \leqslant 1$, $|f(z)| > m$ while $|f(w)| < \epsilon_0 < m/2$. Hence

$$\rho(z, w) \geqslant \left| \frac{f(z) - f(w)}{1 - \overline{f(z)} f(w)} \right| \geqslant |f(z)| - |f(w)| \geqslant m - \epsilon_0 > m/2,$$

so our assertion is true for sets of two points.

Suppose we have demonstrated our assertion for sets of $N - 1$ or fewer points of S. Let T consist of N distinct points of S and let $z_0 \in T$. Let $Q = T \backslash \{z_0\}$. The hypothesis of the theorem guarantees the existence of $f \in H^\infty(D)$ such that $\|f\| \leqslant 1$, $|f(w)| < \epsilon_0$ for all $w \in Q$, and $|f(z_0)| > m$. Define $F: Q \rightarrow \mathbf{C}$ by

$$F(w) = f(w)(1 - w\overline{z_0})/(w - z_0) \quad \text{for} \quad w \in Q;$$

then for all $w \in Q$, $|F(w)| \leqslant 2\epsilon_0/m$ (since $|(1 - w\overline{z_0})/(w - z_0)| = 1/\rho(z_0, w)$, we can apply the case $N = 2$).

By the induction hypothesis, since Q has $N - 1$ points, Q is u.s. with u.s. bound $2/m$. Consequently we can apply the Shapiro and Shields result to obtain $h \in H^\infty(D)$ such that $\|h\| \leqslant 1$ and $h \restriction Q = F$. Define $g \in H^\infty(D)$ by

$$g(z) = \frac{1}{2}\left(f(z) - \frac{z - z_0}{1 - z\overline{z_0}} h(z) \right).$$

Then $\|g\| \leqslant 1$, $g(w) = 0$ for all $w \in Q = T \backslash \{z_0\}$ and $|g(z_0)| > m/2$. This argument will produce such a function for any point in T, hence T is u.s. with bound $2/m$. The induction argument and thus the proof of Theorem 3.5 is complete.

We remark that a theorem similar to Theorem 3.5 is always available whenever uniform separation is necessary and sufficient for interpolation and an explicit expression for the interpolating bound of a sequence, given its u.s. bound, is available.

For sequences in polydisks, Theorem 3.5 has the following consequences.

Let $a = \langle a^{(1)}, a^{(2)}, \cdots, a^{(n)} \rangle \in D^n$ and let $r < 1$; consider the region $D_k(a, r) = \{ z = \langle z^{(1)}, \cdots, z^{(n)} \rangle: \rho_1(z^{(j)}, a^{(j)}) < r$ for all $j = 1, 2, \cdots, n$ except $j = k \}$. $D_k(a, r)$ might be called a "narrow polydisk" in the sense that all but one of its polydisk radii are small. $D_k(a, r)$ contains a copy of D obtained by the embedding

$$\sigma: z \rightarrow \langle a^{(1)}, \cdots, a^{(k-1)}, z, a^{(k+1)}, \cdots, a^{(n)} \rangle,$$

and $D_k(a, r)$ consists of points in D^n whose pseudo-hyperbolic distances from this disk are less than r. Theorem 3.5 implies that a u.s. sequence contained

in a suitable narrow polydisk is interpolating.

COROLLARY 3.6. *Suppose* $S = \{a_i\}_{i=1}^{\infty} \subset D^n$ *is a u.s. sequence with u.s. bound* M, *and suppose there exists* $a \in D^n$ *such that* $S \subset D_k(a, \epsilon_0)$ *where* $\epsilon_0 = (1/4M)\kappa(1/4M)$. *Let* $\pi_j: D^n \longrightarrow D$ *be the projection onto the jth coordinate axis for* $j = 1, 2, \cdots, n$. *Then* $\pi_k(S)$ *is u.s. in* D *with u.s. bound* $4M$ *and consequently* S *is interpolating.*

PROOF. Let $\sigma: z \longrightarrow \langle a^{(1)}, \cdots, a^{(k-1)}, z, a^{(k+1)}, \cdots, a^{(n)} \rangle$ and let $\tau = \sigma \circ \pi_k$ so that $\tau: D^n \longrightarrow \sigma(D)$. Clearly, if $z \in D_k(a, \epsilon_0)$ then $\rho_n(z, \tau(z)) < \epsilon_0$.

S uniformly separated with bound M means that there exist functions $f_1, f_2, \cdots, \in H^{\infty}(D^n)$ such that $\|f_i\| \leqslant 1$ for all i and

$$f_i(a_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1/M & \text{if } i = j. \end{cases}$$

Then since $\rho_n(a_i, \tau(a_i)) < \epsilon_0 < 1/2M$ for all $i, |f_i(\tau(a_i))| > 1/2M$ for all i, and $|f_i(\tau(a_j))| < \epsilon_0 = (1/4M)\kappa(1/4M)$ for all $i \neq j$. So, if $g_i = f_i \circ \sigma, g_i \in H^{\infty}(D), \|g_i\| \leqslant 1$ and $|g_i(\pi_k(a_i))| > 1/2M$ for all i and $|g_i(\pi_k(a_j))| < (1/4M)\kappa(1/4M)$ for all $i \neq j$. Therefore, $\pi_k(S) = \{\pi_k(a_i)\}_{i=1}^{\infty} \subset D$ satisfies the hypothesis of Theorem 3.5 with $m = 1/2M$ and is consequently u.s. with bound $4M$. Finally, $\pi_k(S)$, uniformly separated in D, implies $\pi_k(S)$ is interpolating, so if $\langle \lambda_1, \lambda_2, \cdots \rangle \in l^{\infty}$, there exists $h \in H^{\infty}(D)$ such that $(h \circ \pi_k)(a_i) = h(\pi_k(a_i)) = \lambda_i$. Clearly $h \circ \pi_k \in H^{\infty}(D^n)$, so S is interpolating.

4. **Cartesian products and unions.** Taking Cartesian products of general interpolating sequences preserves the general interpolating property because $H^{\infty}(D^n, A)$ is itself isomorphic to a uniform algebra. (Obviously $H^{\infty}(D^n, A)$ is a Banach algebra and it is clear that if $f \in H^{\infty}(D^n, A), \|f^2\| = \|f\|^2$ since A is a uniform algebra. Therefore, the Gelfand mapping for $H^{\infty}(D^n, A)$ is an isometric isomorphism, and $H^{\infty}(D^n, A)$ is isomorphic to a uniform algebra.)

PROPOSITION 4.1. *If* $S = \{a_i\}_{i=1}^{\infty} \subset D^n$ *is a general interpolating sequence with bound* M, *then* $S \times S \subset D^{2n}$ *is a general interpolating sequence with bound* M^2.

PROOF. Let A be any uniform algebra.

If $\{X_{ij}\}_{i=1; j=1}^{\infty} \subset A$ then for each i, there is $f_i \in H^{\infty}(D^n, A)$ such that $f_i(a_j) = X_{ij}$ where $\|f_i\| \leqslant M \sup_j \|X_{ij}\|$. But we can interpolate into $H^{\infty}(D^n, A)$ i.e. there exists $F: D^n \longrightarrow H^{\infty}(D^n, A)$ such that F is bounded and analytic, $F(a_i) = f_i$ and

$$\|F(z)\| \leqslant M \sup_i \|f_i\| \leqslant M^2 \sup_{i,j} \|X_{ij}\| \quad \text{for each } z \in D^n.$$

Let $G: D^{2n} \to A$ be defined by $G(\langle z, w \rangle) = (F(z))(w)$; then G will be analytic, $\|G(\langle z, w \rangle)\| \leqslant M^2 \sup_{i,j} \|X_{ij}\|$ for all $\langle z, w \rangle \in D^n \times D^n$ and

$$G(\langle a_i, a_j \rangle) = (F(a_i))(a_j) = f_i(a_j) = X_{ij}.$$

The above arguments also show that if $S_1, S_2, \cdots \subset D^n$ are interpolating sequences (w.r.t. A) each with bound M_1 and $T_1, T_2, T_3, \cdots \subset D^n$ are generally interpolating sequences, each with bound M_2, and $S = \{a_i\}_{i=1}^{\infty} \subset D^k$ is a general interpolating sequence with bound M_0, then the sequence $\{\{a_i\} \times S_i\}_{i=1}^{\infty} \subset D^{n+k}$ is interpolating w.r.t. A with bound $M_1 M_0$, and the sequence $\{\{a_i\} \times T_i\}_{i=1}^{\infty} \subset D^{n+k}$ is a general interpolating sequence with bound $M_2 M_0$.

In order to take the union of two interpolating sequences, we simply need a function which vanishes on one sequence and is a nonzero constant on the other. (See the work of Varopoulos [16] for other results on the unions of interpolating sets.)

PROPOSITION 4.2. *If A is a uniform algebra, and Q and R are any two subsets in m_A, $\mathrm{Hk}(Q) \cap \mathrm{Hk}(R) = \varnothing$ implies the existence of elements x and y in A such that*

$$x(Q) = 0, \quad x(R) = 1, \quad and \quad y(Q) = 1, \quad y(R) = 0.$$

PROOF. Let I_Q be the ideal (kernel) generated by Q, i.e. $I_Q = \{x \in A : x(Q) = 0\}$, similarly define I_R, and let I be the ideal generated by $I_Q \cup I_R$. If there is a maximal ideal $m \in m_A$ containing I then $x(m) = 0$ for all $x \in I_Q \cup I_R$. Hence $m \in \mathrm{Hk}(Q) \cap \mathrm{Hk}(R)$ which is impossible. Therefore, $I = A$ and there must exist $x \in I_Q$ and $y \in I_R$ such that $x + y = 1$. Clearly x and y are the desired functions.

In the following theorem and corollaries we re-emphasize that the symbol Cl represents the weak-* closure in the maximal ideal space.

THEOREM 4.3. *If $S \subset D^n$ is an interpolating sequence (w.r.t. C) then* $\mathrm{Cl}(S) = \mathrm{Hk}(S)$.

PROOF. Clearly $\mathrm{Cl}(S) \subset \mathrm{Hk}(S)$. Suppose $z \notin \mathrm{Cl}(S)$. Then there exists a weak-* neighborhood N of z which is disjoint from $\mathrm{Cl}(S)$. N will contain a basic neighborhood of the form $\{w \in m_n : |f_i(w)| < \epsilon$ where $f_i \in H^{\infty}(D^n)$, $f_i(z) = 0$ for $i = 1, 2, \cdots, k\}$. Hence $\mathrm{Cl}(S) = \bigcup_{i=1}^{k} S_i$ where $S_i = \mathrm{Cl}(S) \cap \{w : |f_i(w)| \geqslant \epsilon/2\}$. For each i, $i = 1, 2, \cdots, k$, $1/f_i(S)$ will be a bounded continuous function on the closed subset S_i of the compact Hausdorf space $\mathrm{Cl}(S)$, so $1/f_i$ has a bounded continuous extension to all of $\mathrm{Cl}(S)$; call this extension h_i. S an interpolating sequence implies the existence of $g_i \in H^{\infty}(D^n)$ where

$g_i \upharpoonright S = h_i$, in particular, $g_i \upharpoonright S_i = 1/f_i$. Therefore, $1 - g_i f_i \in H^\infty(D^n)$ and has values 1 at z and 0 on S_i. Finally let $f = \Pi_{i=1}^k (1 - g_i f_i)$, $f \in H^\infty(D^n)$, $f(z) = 1$, and $f(S) = 0$, so $z \notin \mathrm{Hk}(S)$.

COROLLARY 4.4. *If S and T are (general) interpolating sequences in D^n with disjoint weak-* closures in m_n, then $S \cup T$ is a (general) interpolating sequence.*

PROOF. Suppose S and T are interpolating w.r.t. A. By Theorem 4.3, $\mathrm{Hk}(S) \cap \mathrm{Hk}(T) = \emptyset$. Hence by Proposition 4.2 there exist $f, g \in H^\infty(D^n)$ such that $f(S) = 0, f(T) = 1, g(S) = 1$, and $g(T) = 0$. Suppose we want to interpolate X_i at each point a_i of S and Y_i at each point b_i of T, where $\{X_i\}_{i=1}^\infty, \{Y_i\}_{i=1}^\infty \in l^\infty A$.

We know there exists $h_1, h_2 \in H^\infty(D^n, A)$ such that $h_1(a_i) = X_i$ and $h_2(b_i) = Y_i$. So let $h = gh_1 + fh_2$. $h \in H^\infty(D^n, A)$ and $h(a_i) = X_i$ and $h(b_i) = Y_i$ for all i.

We put the above results in a framework which we will find most useful.

PROPOSITION 4.5. *Suppose $m_n = G_1 \cup G_2 \cup \cdots \cup G_m$ where each G_i is a weak-* open subset of m_n, and suppose $S \subset D^n$ is a discrete sequence such that $S \cap G_i$ is an interpolating sequence w.r.t. A for $i = 1, 2, \cdots, m$. Then S is an interpolating sequence w.r.t. A.*

PROOF. We recall that interpolating w.r.t. A automatically implies interpolating w.r.t. C. Also we remark that if Q is an interpolating sequence (w.r.t. C) in D^n, and $T \subset Q$, then there exists $f \in H^\infty(D^n)$ such that $f(T) = 0$ and $f(Q \backslash T) = 1$, hence $\mathrm{Cl}(T) \cap \mathrm{Cl}(Q \backslash T) = \emptyset$.

Let $S_k = S \cap (G_1 \cup G_2 \cup \cdots \cup G_k)$ we show S_k is interpolating w.r.t. A for $k = 1, 2, \cdots, n$. By hypothesis, S_1 is interpolating. Suppose S_{k-1} is interpolating w.r.t. A. Let $G = G_1 \cup G_2 \cup \cdots \cup G_{k-1}$ so that $S_{k-1} = S \cap G$ and let $T = S \cap (G_k \backslash G)$, then S_{k-1} and T will be disjoint interpolating sequences w.r.t. A. We show $\mathrm{Cl}(S_{k-1}) \cap \mathrm{Cl}(T) = \emptyset$ in which case Corollary 4.4 will imply $S_k = S_{k-1} \cup T$ is an interpolating sequence w.r.t. A.

For any j, $(S_{k-1} \cap G_j) \cup (T \cap G_j) \subset S \cap G_j$ and by hypothesis $(S_{k-1} \cap G_j) \cup (T \cap G_j)$ is interpolating, so by the remark at the beginning of this proof $\mathrm{Cl}(S_{k-1} \cap G_j) \cap \mathrm{Cl}(T \cap G_j) = \emptyset$. Now suppose $a \in \mathrm{Cl}(S_{k-1}) \cap \mathrm{Cl}(T)$, then $a \in m_n$ implies $a \in G_j$ for some j and since G_j is an open set, $a \in \mathrm{Cl}(S_{k-1} \cap G_j) \cap \mathrm{Cl}(T \cap G_j) = \emptyset$. Hence $\mathrm{Cl}(S_{k-1}) \cap \mathrm{Cl}(T) = \emptyset$ as desired.

We remark that the above proposition is clearly true if we replace the phrase "interpolating w.r.t. A" with "general interpolating". Finally, we give a set of concrete conditions under which we will be able to apply Proposition 4.5.

LEMMA 4.6. *Suppose we are given* $F_1, F_2, \cdots, F_k$ *where each* F_j *is a finite collection of functions in* $H^\infty(D^n)$. *Fix two positive constants* $0 < r < t$ *and let* $F_j = \{z \in D^n: |f(z)| \leq r$ *for all* $f \in F_j\}$ *and* $E_j = \{z \in D^n: |f(z)| < t$ *for all* $f \in F_j\}$. *Then there exists* $G_0, G_1, \cdots, G_k$ *open subsets of* m_n *such that*

(1) $G_0 \cap D^n = D^n \backslash (\bigcup_{j=1}^k F_j)$ *and* $G_j \cap D^n = E_j$ *for* $j = 1, 2, \cdots, k$,

(2) $m_n = \bigcup_{j=0}^k G_j$.

PROOF. Let

$$G_0 = m_n \backslash \mathrm{Cl}\left(\bigcup_{j=1}^k F_j\right), \qquad G_j = m_n \backslash \mathrm{Cl}(D^n \backslash E_j) \quad \text{for } j = 1, 2, \cdots, k;$$

then clearly $G_0 \cap D^n = D^n \backslash (\bigcup_{j=1}^k F_j)$ and $G_j \cap D^n = E_j$. If $\beta \in m_n \backslash (\bigcup_{j=0}^k G_j)$ then

$$\beta \in \mathrm{Cl}\left(\bigcup_{j=1}^k F_j\right) \cap \left(\bigcap_{j=1}^k \mathrm{Cl}(D^n \backslash E_j)\right).$$

$\beta \in \mathrm{Cl}(\bigcup_{j=1}^k F_j) = \bigcup_{j=1}^k \mathrm{Cl}(F_j)$ implies $\beta \in \mathrm{Cl}\, F_j$ for some fixed j, say $j = 1$; consequently, $|f(\beta)| \leq r$ for all $f \in F_1$. This implies the existence of a neighborhood V of β in m_n on which $|f| < (r + t)/2$ for all $f \in F_1$. We also have $\beta \in \mathrm{Cl}(D^n \backslash E_1)$ which implies $V \cap (D^n \backslash E_1)$ is not empty. But, $z \in D^n \backslash E_1$ implies there is some $f \in F_1$ such that $|f(z)| \geq t$, so $z \in V \cap (D^n \backslash E_1)$ implies $t \leq |f(z)| < (r + t)/2 < t$ which is impossible. Therefore, β cannot exist and $m_n = \bigcup_{j=0}^k G_j$.

5. **Interpolating sequences in wedges.** Our first example of an α-separated sequences is given by the following proposition.

PROPOSITION 5.1. *If* $S = \{a_i\}_{i=1}^\infty \subset D$ *is a u.s. sequence and there exists* $\tau \geq 0$ *such that* $S \subset \{z \in D: (1 - |z|^2)/|1 - z^2| > \tau\} \cap \{z \in D: \mathrm{Im}(z) \geq 0\}$. *Then* S *is* α-separated.

PROOF. S uniformly separated implies that

$$\rho(a_i, a_j) = |(a_i - a_j)/(1 - a_i \bar{a}_j)| \geq p \quad \text{for some } p,$$

and $\mathrm{Inf}_i \Pi_{j=1; j \neq i}^\infty \rho(a_i, a_j) > 0$; so there exists M such that $\Sigma_{j=1}^\infty 1 - \rho(a_i, a_j) \leq M$ for all i.

Let

$$\alpha_j(z) = \frac{z - a_j}{1 - z\bar{a}_j} \frac{z - \bar{a}_j}{1 - za_j}.$$

Then

$$\alpha_j \in H^\infty(D), \quad \|\alpha_j\| \leqslant 1, \quad \alpha_j(a_j) = 0$$

and

$$|\alpha_j(a_i)| = \rho(a_j, a_i)\rho(\bar{a}_j, a_i) \geqslant p^2.$$

(This last inequality follows from the fact that since $\operatorname{Im}(a_i) \geqslant 0$ and $\operatorname{Im}(a_j) \geqslant 0$, $\rho(\bar{a}_j, a_i) \geqslant \rho(a_j, a_i)$.) Standard arguments show that

$$\sum_{j=1}^{\infty} |1 - \alpha_i(z)| = \sum_{i=1}^{\infty} \frac{|(1 - z^2)|(1 - |a_j|^2)}{|(1 - a_j z)(1 - \bar{a}_j z)|}$$

converges uniformly on compact disks in D.

Finally,

$$|1 - \alpha_j(a_i)| = \frac{|(1 - a_i^2)|(1 - |a_j|^2)}{|(1 - a_j a_i)(1 - \bar{a}_j a_i)|}$$

$$= \frac{|1 - a_i^2|}{1 - |a_i|^2} \left| \frac{1 - a_j \bar{a}_i}{1 - a_j a_i} \right| \frac{(1 - |a_i|^2)(1 - |a_j|^2)}{|(1 - a_j \bar{a}_i)(1 - \bar{a}_j a_i)|}$$

$$= \frac{|1 - a_i^2|}{1 - |a_i|^2} \left| \frac{1 - a_j \bar{a}_i}{1 - a_j a_i} \right| (1 - \rho(a_j a_i)^2)$$

$$\leqslant 2 \left| \frac{1 - a_j \bar{a}_i}{1 - a_j a_i} \right| \frac{|1 - a_i^2|}{1 - |a_i|^2} (1 - \rho(a_j a_i)).$$

A fairly direct calculation again shows that if $\operatorname{Im} a_j \geqslant 0$ and $\operatorname{Im} a_i \geqslant 0$ then $|(1 - a_j \bar{a}_i)/(1 - a_j a_i)| \leqslant 1$ so

$$\sum_{j=1; j \neq i}^{\infty} |1 - \alpha_j(a_i)| \leqslant 2 \frac{|1 - a_i^2|}{1 - |a_i|^2} \sum_{j=1; j \neq i}^{\infty} (1 - \rho(a_i, a_j)) \leqslant \frac{2}{\tau} M.$$

Therefore, S is α-separated.

COROLLARY 5.2. *If* $S = \{a_i\}_{i=1}^{\infty} \subset D$ *is u.s. and there exists* $\tau > 0$ *such that* $S \subset \{z \in D: (1 - |z|^2)/|1 - z^2| > \tau\}$ *then* S *is a general interpolating sequence.*

PROOF. By Proposition 5.1, $S_1 = S \cap \{z \in D: \operatorname{Im}(z) \geqslant 0\}$ and $S_2 = S \cap \{z \in D: \operatorname{Im}(z) < 0\}$ are α-separated and hence general interpolating sequences. But $S = S_1 \cup S_2$ is u.s. in D and consequently, by Carleson's theorem, S is interpolating. Since S_1 and S_2 are disjoint subsequences of an interpolating sequence, $\operatorname{Cl}(S_1) \cap \operatorname{Cl}(S_2) = \varnothing$ and we can apply Corollary 4.4.

Our goal is to generalize Corollary 5.2 to a number of different contexts. We begin by considering the region

$$W_\tau = \{z \in D: (1 - |z|^2)/(|1 - z^2|) \geqslant \tau\}.$$

Corollary 5.2 tells us that if $S \subset W_\tau$ is a u.s. sequence then it is a general interpolating sequence. W_τ is readily seen to be the region in D lying between the arcs of two circles which intersect at ± 1. (The circles have centers at $\pm i\tau\sqrt{1 - \tau^2}$ and both have radius $1/\sqrt{1 - \tau^2}$.) In general, we make the following definition.

DEFINITION 5.3. A wedge in D is the region in D lying between the arcs of two circles which intersect twice on the boundary of D.

Since linear fractional mappings send circles into circles, it is always possible to find a conformal self mapping of D which maps any wedge into a subset of a wedge of the type $W_\tau = \{z \in D: (1 - |z|^2)/|1 - z^2| \geq \tau\}$. This fact immediately yields the following generalization of Corollary 5.2.

COROLLARY 5.4. *If $S \subset D$ is a u.s. sequence contained in a wedge, then S is a general interpolating sequence.*

DEFINITION 5.5. We will call a region $R \subset D$ wedge-like if R is contained in a finite union of wedges.

COROLLARY 5.6. *If $S \subset D$ is a u.s. sequence contained in a wedge-like region of D, then S is a general interpolating sequence.*

PROOF. Since S is contained in a wedge-like region, it has only a finite number of limit points (in the usual topology for C) on T^1. Call these limit points $\phi_1, \cdots, \phi_k$. It is clear that there exist constants $0 < s < t < 1$ such that for $j = 1, 2, \cdots, k$, $S \cap \{z \in D: |z - \phi_j| < t\}$ is contained in a wedge and is therefore a general interpolating sequence. By Lemma 4.6 there exist $G_0, G_1, \cdots, G_k$ open sets of m_1 such that $G_0 \cap D = D\backslash(\bigcup_{j=1}^k \{z \in D: |z - \phi_j| \leq s\})$ while $G_j \cap D = \{z \in D: |z - \phi_j| < t\}$ and $m_1 = \bigcup_{j=0}^k G_j$. But $S \cap G_0$ will be a finite number of points hence general interpolating, and as we remarked above $S \cap G_j$ will be general interpolating for $j = 1, 2, \cdots, k$. The corollary then follows from Proposition 4.5.

We can generalize the above results to polydisks with the help of the constructive techniques of §§3 and 4.

DEFINITION 5.7. A wedge W in D^n is the Cartesian product of n wedges in D.

If $W_\tau = \{z \in D: (1 - |z^2|)/(|1 - z^2|) \geq \tau\}$ then it is easy to see that after application of a coordinate-wise conformal self-mapping of D^n onto itself, any wedge in D^n can be considered a subset of $W_\tau \times W_\tau \times \cdots \times W_\tau = W_\tau^n$ for an appropriate τ.

DEFINITION 5.8. A subset $\Omega \subset D^n$ is a near-wedge if there exists a wedge $W \subset D^{n-1}$ such that $\Omega = D \times W$. (If $n = 1$, for consistency, we will say that a near-wedge is just all of D.)

The generalization we are after is:

THEOREM 5.9. *Suppose $S \subset D^n$ is a uniformly separated sequence, then*

(a) *if S is contained in a wedge in D^n, then S is a general interpolating sequence,*

(b) *if S is contained in a near-wedge in D^n, S is an (ordinary) interpolating sequence.*

(*Note.* If $n = 1$, Theorem 5.9 is just Carleson's theorem and Corollary 5.4.)

PROOF. Assume m is a u.s. bound for S. By virtue of the remarks following the definition of a wedge in D^n, we may assume all wedges are of the form, $W_\tau \times W_\tau \times \cdots \times W_\tau$. The proof progresses through a series of lemmas. If S is a u.s. sequence in D^n and $0 < \epsilon < 1$, define the ϵ-neighborhood of S as

$$U(S, \epsilon) = \{z \in D^n : \rho_n(z, a) < \epsilon \text{ for some } a \in S\}.$$

$U(S, \epsilon)$ is the union of an infinite sequence of small polydisks. Since S is u.s., if we always assume $\epsilon < 1/2m$, where m is a u.s. bound for S, then it is clear that the polydisks of $U(S, \epsilon)$ will all be disjoint.

LEMMA 5.10. *For a given wedge $W \subset D^n$ there exist constants δ, ϵ_0, and $M_0 > 0$ such that for any $\epsilon < \epsilon_0$ there exist general interpolating sequences $S_1, S_2, \cdots, S_N$ where N depends only on ϵ, such that each S_i has general interpolating bound M_0, and the ϵ-neighborhoods of $S_1, S_2, \cdots, S_N$ cover W (i.e. $W \subset \bigcup_{i=1}^{N} U(S_i, \epsilon)$). Moreover, each S_i is of the form $S_i = R_1 \times R_2 \times \cdots \times R_n$ where each R_j is u.s. in D with u.s. bound $1/\delta$.*

PROOF. We begin by considering $W_\tau = \{z \in D : (1 - |z^2|)/(|1 - z^2|) \geq \tau\}$. Fix α, $3/4 < \alpha < 1$, say $\alpha = 4/5$, and consider $\sigma: D \longrightarrow D$ by $\sigma(z) = (z + \alpha)/(1 + \alpha z)$. σ generates an infinite cyclic group of automorphism of D with composition as the group operation $(\sigma^{-1}(z) = z - \alpha/1 - \alpha z)$. It is easy to verify that $\sigma(W_\tau) = W_\tau$, so $\sigma^k(W_\tau) = W_\tau$ for $k = \pm1, \pm2, \cdots$. It is also clear that σ^k leaves no point of W_τ fixed for $k = \pm1, \pm2, \cdots$. Let $\gamma_0 = \{z \in W_\tau : \text{Re } z = 0\}$ and $\gamma_1 = \sigma(\gamma_0)$ and let W_0 be the region of W_τ bounded on the left by γ_0 and on the right by γ_1. We will assume W_0 contains γ_0 but not γ_1. Define $W_k = \sigma^k(W_0)$, $k = \pm1, \pm2, \cdots$. It is easy to see that $\{W_k\}_{k=-\infty}^{\infty}$ is a collection of disjoint subsets of W_τ and $W_\tau = \bigcup_{k=-\infty}^{\infty} W_k$.

Now suppose $z \in W_0$, let $z_0 = z$ and $z_k = \sigma^k(z) \in W_k$ for $k = \pm1, \pm2, \cdots$. Then if $k > 0$, one verifies that

$$\frac{1 - |z_k|}{1 - |z_{k-1}|} \leq 2 \frac{1 - |z_k|^2}{1 - |z_{k-1}|^2} = 2 \frac{1 - |\sigma(z_{k-1})|^2}{1 - |z_{k-1}|^2}$$

$$\leq 2(1 - |\alpha|^2) = 2(1 - 16/25) = 18/25 < 1.$$

So $z_0, z_1, z_2, \cdots$ is an exponential sequence in D, hence it is u.s. with u.s.

bound derivable from the ratios $(1 - |z_k|)/(1 - |z_{k-1}|) \leqslant 18/25$. Similarly, $z_{-1}, z_{-2}, \cdots$ is u.s. in D, and it follows that $R_z = \{\cdots z_{-2}, z_{-1}, z_0, z_1, z_2, \cdots\}$ will be u.s. with u.s. bound depending only on the choice of $\alpha = 4/5$. Hence, there are fixed constants, δ, M such that any sequence of the form R_z will be u.s. with bound $1/\delta$ and consequently a general interpolating sequence with bound M where M depends only on δ and τ.

Let ϵ_0 be small enough so that for any u.s. sequences $Q \subset D$ with u.s. bound $1/\delta$, the disks of $U(Q, \epsilon_0)$ will be disjoint. We repeat that M, δ, ϵ_0 depend only on W_τ and the choice of $\alpha = 4/5$. Suppose $\epsilon < \epsilon_0$, then we can cover W_0 by a finite number, say r, pseudo-hyperbolic disks, $D_1, D_2, \cdots,$ D_r of ρ-radius ϵ with centers in W_0. Thus, $W_0 \subset \bigcup_{i=1}^r D_i$ and consequently $W_k \subset \bigcup_{i=1}^r \sigma^k(D_i)$. If $a_1, a_2, \cdots, a_r$ are the respective centers of $D_1, D_2,$ $\cdots, D_r$ we let $R_i = R_{a_i} = \{\sigma^k(a_i)\}_{k=-\infty}^\infty$. Then, the sequences $R_1, R_2, \cdots,$ R_r will be general interpolating sequences in D, all with bound M, and $\bigcup_{i=1}^r U(R_i, \epsilon) \supset W_\tau$.

Finally, if $S = R_{i_1} \times R_{i_2} \times \cdots \times R_{i_n}$ is the Cartesian product of any n of the R_i's, S will be a general interpolating sequence in D^n with general interpolating bound $M_0 = M^n$. There are exactly $N = r^n$ such sequences, call them $S_1, S_2, \cdots, S_N$. Clearly $W_\tau \times W_\tau \times \cdots \times W_\tau \subset \bigcup_{i=1}^N U(S_i, \epsilon)$. This completes the proof of Lemma 5.10.

LEMMA 5.11. *Suppose* $T \subset D^n$ *is a u.s. sequence of the form* $T = R_1 \times R_2 \times \cdots \times R_n$ *where* $R_k \subset D$ *is a u.s. sequence with u.s. bound* $1/\delta$ *for* $k = 1, 2, \cdots, n$. *Let* $\eta < \delta/2$, *then if* $\epsilon < \frac{1}{2}\eta^3$, *there exist* $f_1, f_2, \cdots,$ $f_n \in H^\infty(D^n)$ *such that* $U(T, \epsilon) \subset \{z \in D^n : |f_i(z)| < \eta^3, i = 1, 2, \cdots, n\} \subset$ $U(T, \eta)$.

PROOF. Let $B_k \in H^\infty(D)$ be the Blaschke product with zeros precisely at R_k. Then it can be shown (see for example [9, Lemmas 4.2 and 4.3]) that if η is chosen such that $\eta < \delta/2$, then

$$\{z \in D : |B_k(z)| < \eta^3\} \subset \{z \in D : \rho_1(z, a) < \eta \text{ for some } a \in R_i\}.$$

Letting $f_k \in H^\infty(D^n)$ be defined by $f_k(\langle z^{(1)}, \cdots, z^{(n)} \rangle) = B_k(z^{(k)})$, the lemma follows immediately.

LEMMA 5.12. *Suppose* S *is a u.s. sequence of u.s. bound* m, T *is a general interpolating sequence of bound* M, *and suppose* $\eta < \text{Min } \{1/2m, 1/4M\}$ *then* $S \cap U(T, \eta)$ *is a general interpolating sequence.*

PROOF. $U(T, \eta)$ consists of a disjoint union of polydisks of the form $D_a = \{z \in D^n : \rho(z, a) < \eta\}$ where $a \in T$. By the choice of $\eta < 1/2m$, there exists at most one element of S in D_a, hence $S \cap U(T, \eta)$ can be obtained by jiggling each point of a subsequence of T a (pseudo-hyperbolic) distance

$\leqslant 1/4M$. Therefore, by Theorem 3.4, $S \cap U(T, \eta)$ is a general interpolating sequence.

PROOF OF THEOREM 5.9(a). Let ϵ_0, δ, M_0 be the constants of Lemma 5.10 which, we recall, depend only on W. Let $\eta < \text{Min } \{\tfrac{1}{2}\delta, 1/2m, 1/4M_0, \epsilon_0\}$ and let $\epsilon \leqslant \tfrac{1}{2}\eta^3$. Then by Lemma 5.10 there exist N sequences $T_1, T_2, \cdots, T_N$ (where N depends only on ϵ) such that $W \subset \bigcup_{i=1}^N U(T_i, \epsilon)$, each T_i is a general interpolating sequence of bound M_0, and is the Cartesian product of n u.s. sequences in D, each with u.s. bound $1/\delta$.

By Lemma 5.11, for each T_i there is a family, $\vec{F}_i$, of n functions in $H^\infty(D^n)$ such that

$$U(T_i, \epsilon) \subset \{z \in D^n : |f(z)| < \eta^3 \text{ for all } f \in F_i\} \subset U(T_i, \eta).$$

Applying Lemma 4.6, there exists $G_0, G_1, G_2, \cdots, G_N$ open subsets of m_n such that $m_n = \bigcup_{i=0}^N G_i$ and

$$G_0 \cap D^n = D^n \Big\backslash \bigcup_{i=1}^N \{z \in D^n : |f(z)| \leqslant \epsilon \text{ for all } f \in F_i\}$$

$$\subset D^n \Big\backslash \bigcup_{i=1}^N U(T_i, \epsilon) \subset D^n \backslash W,$$

while

$$G_i \cap D^n = \{z \in D^n : |f(z)| < \eta^3 \text{ for all } f \in F_i\} \subset U(T_i, \eta)$$

for all $i = 1, 2, \cdots, N$. Moreover,

$$S \cap G_0 = S \cap G_0 \cap D^n \subset S \cap (D^n \backslash W) = \varnothing$$

while

$$S \cap G_i = S \cap G_i \cap D^n \subset S \cap U(T_i, \eta)$$

which, by Lemma 5.12 and the choice of η is a general interpolating sequence for $i = 1, 2, \cdots, N$. Therefore we can apply Proposition 4.5 to demonstrate that S is a general interpolating sequence. This completes the proof of part (a) of Theorem 5.9.

The proof of part (b) requires much of the machinery set up for part (a), so to avoid confusing changes in notation, we assume $S \subset W \times D \subset D^{n+1}$ is a u.s. sequence in D^{n+1} of u.s. bound m, where W is the same wedge we used in part (a). Under these assumptions, the constants ϵ_0, M_0, δ and the sequences of Lemma 5.10 and the constants and functions of Lemma 5.11 can be used again. We must, however, modify Lemma 5.12. Using the notation of §3, we recall that a u.s. sequence in D with u.s. bound $1/\delta$ is interpolating, with interpolating bound $1/\kappa(\delta)$. (This is the Shapiro and Shields estimate [14] for Carleson's theorem.)

LEMMA 5.13. *Suppose* $S \subset D^{n+1}$ *is u.s. with u.s. bound* m, *and* $T \subset D^n$ *is a general interpolating sequence with bound* M. *Let* $\epsilon < (1/4mM)\kappa(1/4m)$ *and let* $U'(T, \epsilon) = U(T, \epsilon) \times D$. *Then* $S \cap U'(T, \epsilon)$ *is an interpolating sequence.*

PROOF. $U'(T, \epsilon)$ consists of "narrow polydisks" of the form

$$P'_a = \{\langle z, w\rangle \in D^n \times D: \rho_n(z, a) < \epsilon\} \quad \text{for} \quad a \in T.$$

For each $a \in T$, $S \cap P'_a$ satisfies the hypothesis of Corollary 3.6 and therefore $\pi_{n+1}(S \cap P'_a)$ is u.s. in D with u.s. bound $4m$. So by the Shapiro and Shields estimate, $\pi_{n+1}(S \cap P'_a)$ is interpolating in D with bound $1/\kappa(1/4m)$. By the remarks following Proposition 4.1, since T is a general interpolating sequence with bound M_0, $Q = \{a \times \pi_{n+1}(S \cap P'_a): a \in T\}$ will be an interpolating sequence with interpolating bound $M/\kappa(1/4m)$. If $s \in S \cap U'(T, \epsilon)$, then $s \in S \cap P'_a$ for some $a \in T$, and

$$\langle a, \pi_{n+1}(s)\rangle \in a \times \pi_{n+1}(S \cap P'_a) \subset Q$$

and

$$\rho_{n+1}(s, \langle a, \pi_{n+1}(s)\rangle) < \epsilon < (1/4M)\kappa(1/4m).$$

Moreover, since S is a u.s. sequence with bound m, $\langle a, \pi_{n+1}(s)\rangle$ will have to be the only point in Q which is within a pseudo-hyperbolic distance $\epsilon < 1/4m$ of s. Therefore, the points of $S \cap U'(T, \epsilon)$ can be obtained by jiggling a subsequence of the points of Q a pseudo-hyperbolic distance less than $\epsilon < (1/4M)\kappa(1/4m)$, where $M/\kappa(1/4m)$ is an interpolating bound for Q. Consequently, we can apply Theorem 3.3, and it follows that S is an interpolating sequence. This proves Lemma 5.13.

PROOF OF THEOREM 5.9(b). The rest of the proof of part (b) of Theorem 5.9 mimics the final part of the proof of part (a). We have $S \subset W \times D$ is a u.s sequence with u.s. bound m. Let ϵ_0, δ, M_0 be the constants of Lemma 5.10, let $\eta < \text{Min} \{\epsilon_0, 1/2\delta, (1/4mM_0)\kappa(1/4m)\}$, and choose $\epsilon < 1/2\eta^3$. Then by Lemma 5.10 there exist N sequences $T_1, T_2, \cdots, T_N$ such that $W \times D \subset \bigcup_{i=1}^N U'(T_i, \epsilon)$ where each T_i is a general interpolating sequence, of bound M_0, and is equal to the Cartesian product of n u.s. sequences in D, each with u.s. bound $1/\delta$. By Lemma 5.11, for each T_i, we have a family F_i of n functions in $H^\infty(D^{n+1})$ such that

$$U'(T_i, \epsilon) \subset \{z \in D^{n+1}: |f(z)| < \eta^3 \text{ for all } f \in F_i\} \subset U'(T_i, \eta).$$

Applying Lemma 4.6, we get $H_0, H_1, \cdots, H_N$ open subsets of m_{n+1} such that $m_{n+1} = \bigcup_{i=0}^N H_i$ and

$$H_0 \cap D^{n+1} = D^{n+1} \Big\backslash \bigcup_{i=1}^{N} \{z \in D^{n+1}: |f(z)| \leqslant \epsilon \text{ for all } f_i \in F_i\}$$

$$\subset D^{n+1} \Big\backslash \bigcup_{i=1}^{N} U'(T_i, \epsilon) \subset D^{n+1} \backslash (W \times D)$$

and

$$H_i \cap D^{n+1} = \{z \in D^{n+1}: |f(z)| < \eta^3 \text{ for all } f_i \in F_i\}$$

$$\subset U'(T_i, \eta) \quad \text{for } i = 1, 2, \cdots, N.$$

Moreover

$$S \cap H_0 \subset S \cap (D^{n+1} \backslash W \times D) = \emptyset$$

while

$$S \cap H_i \subset S \cap U'(T_i, \eta)$$

which, by Lemma 5.13 and the choice of η, is an interpolating sequence for $i = 1, 2, \cdots, N$. We can therefore apply Proposition 4.5 from which it follows that S is an interpolating sequence. This completes the proof of Theorem 5.9.

We can extend the results of Theorem 5.9 in a number of ways.

DEFINITION 5.14. A region $V \subset D^n$ will be called wedge-like if it contained in a finite union of wedges in D^n. A region $V' \subset D^n$ will be called near wedge-like if $V' = D \times V$ where V is a wedge-like region in D^{n-1}.

COROLLARY 5.15. *Suppose S is a u.s. sequence in D^n, then*

(a) *if S is contained in a wedge-like region of D^n, it is a general interpolating sequence,*

(b) *if S is contained in a near wedge-like region of D^n, it is an interpolating sequence (w.r.t. $\mathbf{C}$).*

Finally, we point out that a careful examination of the proof of part (a) of Theorem 5.9 reveals that the hypothesis that S is u.s. was used only to show that $S \cap D_a$ contained at most one point, where D_a was a polydisk of small pseudo-hyperbolic radius. Clearly, a much weaker hypothesis than uniform separation will suffice.

THEOREM 5.16. *If $S \subset D^n$ is a sequence with the property that* Inf $\{\rho(s, t): s, t \in S, s \neq t\} > 0$ *and S is continued in a wedge-like region in D^n, then S is a general interpolating sequence.*

(In 1971, Tse Kam-fook [15] pointed out that pointwise separation in the pseudo-hyperbolic metric is sufficient for interpolation (w.r.t. $\mathbf{C}$) inside a wedge in D. Theorem 5.16 is a broad generalization of this observation.)

6. **Examples of α-separated sequences.** We give a few examples of sequences (primarily in D) which satisfy Definition 2.4 and are therefore general interpolating sequences. One of the examples will show that α-separation is not necessary for general interpolation, at least for sequences in D^n where $n > 1$. When $n = 1$, it is not clear whether there is a distinction between uniformly separated, α-separated, and general interpolating sequences.

We first give a very strong condition under which sequences are α-separated.

DEFINITION 6.1. A sequence $\{a_i\}_{i=1}^{\infty} \subset D^n$ is Π-separated if there exist $f_i \in H^{\infty}(D^n)$, $\|f_i\| \leqslant 1$ and

$$f_i(a_j) = \begin{cases} 0 & \text{if } i \neq j \\ \delta_i > 0 & \text{if } i = j \end{cases} \quad \text{and} \quad \prod_{i=1}^{\infty} \delta_i = p > 0.$$

(This condition is suggested by a more restrictive condition which A. Dufresnoy showed was sufficient for interpolation w.r.t. **C**.)

PROPOSITION 6.2. *A Π-separated sequence in D^n is α-separated.*

PROOF. Let $\{a_i\}_{i=1}^{\infty}$ be Π-separated and let f_i, δ_i, p be as in the definition.

Let

$$g_i(z) = (\delta_i - f_i(z))/(1 - \delta_i f_i(z)).$$

Then $g_i \in H^{\infty}(D^n)$, $\|g_i\| \leqslant 1$, $g_i(a_i) = 0$ and $g_i(a_j) = \delta_i \geqslant p$ if $i \neq j$. Consequently

$$\sum_{i=1}^{\infty} |1 - g_i(a_j)| = 1 + \sum_{i=1; i \neq j}^{\infty} (1 - \delta_i) \leqslant 1 + \sum_{i=1}^{\infty} (1 - \delta_i) \leqslant \ln \frac{1}{p} + 1.$$

Now we can assume $\delta_1 \neq 1$, and if $i \neq 1, f_i(a_1) = 0$. This implies $|f_i(z)| \leqslant \rho(z, a_i)$, so for each compact polydisk $\text{Cl}(D^n(r)) \subseteq D^n$ there is a constant $R_r = \text{Max } \{\rho(z, a_1) | z \in \text{Cl}(D^n(r))\}$ so that $|f_i(z)| \leqslant R_r$. Consequently,

$$|1 - g_i(z)| = \left| \frac{1 - \delta_i f_i(z) - \delta_i + f_i(z)}{1 - \delta_i f_i(z)} \right| = (1 - \delta_i) \frac{|1 + f_i(z)|}{|1 - \delta_i f_i(z)|}$$

and $|1 - g_i(z)| \leqslant (2/1 - R_r)(1 - \delta_i)$ for all z in $\text{Cl}(D^n(r))$ and $i > 1$. Therefore

$$\sum_{i=1}^{\infty} |1 - g_i(z)| \leqslant 2 \left(1 + \frac{1}{1 - R_r} \sum_{i=2}^{\infty} (1 - \delta_i) \right) \leqslant 2 \left(1 + \frac{1}{1 - R_r} \ln \frac{1}{p} \right)$$

for all $z \in \text{Cl}(D^n(r))$. Therefore the g_i's satisfy all the conditions necessary to make $\{a_i\}_{i=1}^{\infty}$ α-separated.

It should be noted that although one can construct Π-separated sequences, Π-separation is a very strong condition since the pseudo-hyperbolic distances between points in the sequence must be rapidly increasing. In fact, the minimum

pseudo-hyperbolic distance, d_i, between a_i and any other point of the sequence, must tend to one at a rate which is at least fast enough to insure $\Pi_{i=1}^{\infty} d_i > 0$. Many interpolating sequences in D do not have such increasing distances between points. For example, even sequences tending exponentially to the boundary are not necessarily Π-separated. Indeed, if $a_n = 1 - r^n$ where $0 < r < 1$, then

$$\rho(a_n, a_{n+1}) = \frac{(1 - r^{n+1}) - (1 - r^n)}{1 - (1 - r^{n+1})(1 - r^n)} = \frac{r^n - r^{n+1}}{r^n + r^{n+1} - r^{2n+1}} = \frac{1 - r}{1 + r + r^{n+1}}$$

so $\rho(a_n, a_{n+1}) \rightarrow (1 - r)/(1 + r)$ as $n \rightarrow \infty$.

We mentioned that the result of Corollary 5.16 in the case of sequences in D which were interpolating (w.r.t. $\mathbf{C}$) was obtained by Tse Kam-fook. His observation was based on an earlier result of D. Newman [12] which stated that if $S = \{a_n\}_{n=1}^{\infty} \subset D$ converges nontangentially to the point 1 and $\operatorname{Inf} \rho(a_j, a_j) > 0$, then a necessary and sufficient condition that S be an interpolating sequence (and hence, as we have seen, a general interpolating sequence) is

(N)   The set $\{(1 - a_n)/(1 - a_m): a_n, a_m \in S, n \neq m\}$
          does not have 1 as a limit point.

By $a_n \rightarrow 1$ nontangentially, we mean that $a_n \rightarrow 1$ inside a wedge. We say that $a_n \rightarrow 1$ tangentially if $\{a_n\}_{n=1}^{\infty}$ is contained in no single wedge, i.e.

$$\operatorname{Inf}_{n} \frac{1 - |a_n|^2}{|1 - a_n^2|} = 0.$$

We show that under somewhat stronger tangential behavior (N) will be sufficient to guarantee general interpolation.

PROPOSITION 6.3. *If*

$$S = \{a_n\}_{n=1}^{\infty} \subset D \quad and \quad \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - a_n^2|} = M < \infty$$

and $\{(1 - a_n)/(1 - a_m): a_n, a_m \in S; n \neq m\}$ *does not have* 1 *as a limit point, then* S *is* α*-separated and consequently* S *is a general interpolating sequence.*

PROOF. Let

$$r = \operatorname{Inf}\{|1 - (1 - a_n)/(1 - a_m)|: a_n, a_m \in S, n \neq m\};$$

then $r > 0$. Let

$$\alpha_n(z) = \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z}.$$

If $n \neq m$, suppose $|(1 - a_n)/(1 - a_m)| \leq 1$; then

$$|\alpha_n(a_m)| = |\alpha_m(a_n)| = \left|\frac{a_n - a_m}{1 - a_n \bar{a}_m}\right|$$

$$= \left|\frac{(1 - a_m) - (1 - a_n)}{1 - a_m}\right| \left|\frac{1 - a_m}{(1 - a_m) + a_m(1 - \bar{a}_n)}\right|$$

$$= \left|1 - \frac{1 - a_n}{1 - a_m}\right| \left|\frac{1}{1 + a_m(1 - \bar{a}_n)/(1 - a_m)}\right|$$

$$\geqslant r \frac{1}{1 + |(1 - \bar{a}_n)/(1 - a_m)|} \geqslant \frac{r}{2}.$$

So condition (1) and (2) of the definition of α-separation are met. The hypothesis guarantees the convergence of the Blaschke product with zeroes at $\{a_n\}_{n=1}^{\infty}$, so condition (4) is also met.

Finally, if $n \neq m$, $r \leqslant 1 - (1 - a_m)/(1 - a_n) = (a_m - a_n)/(1 - a_n)$, so

$$|1 - \alpha_n(a_m)| = \frac{1 - |a_n|}{|1 - \bar{a}_n a_m|} |\,|a_n| + \bar{a}_n a_m|$$

$$\leqslant 2 \frac{1 - |a_n|}{|1 - a_n|} \left|\frac{1 - a_n}{a_m - a_n}\right| \left|\frac{a_m - a_n}{1 - \bar{a}_n a_m}\right|$$

$$= \frac{2}{r} \frac{1 - |a_n|^2}{|1 - a_n^2|} \frac{|1 + a_n|}{1 + |a_n|} \leqslant \frac{2}{r} \frac{1 - |a_n|^2}{|1 - a_n^2|}$$

so $\Sigma_{n=1;\, n \neq m}^{\infty} |1 - \alpha_n(a_m)| \leqslant 2M/r$ and condition (3) is met; hence S is α-separated.

An example of a sequence satisfying the hypothesis of Proposition 4.18 is given by the following "doubly exponential" sequence:

$$S = \{a_n\}_{n=1}^{\infty} \quad \text{where} \quad a_n \in \{z: 1 - |z| = r_1^n\} \cap \{z: |1 - z| = r_2^n\}$$

$$\text{where} \quad 0 < r_1 < r_2 < 1.$$

On the other hand, the sequence, $T = \{b_n\}_{n=1}^{\infty}$ where

$$b_n = \{z: 1 - |z| = 1/n^4\} \cap \{z: |1 - z| = 1/n\} \cap \{z: \text{Im } z \geqslant 0\}$$

is an example of an α-separated sequence which satisfies all the hypotheses of Proposition 6.3 except condition (N).

We note that Corollary 5.4 and Proposition 6.3 exhibit u.s. sequences in D with extreme tangential and nontangential behavior as general interpolating sequences.

In 1956, A. G. Naftalevitch described a method of constructing u.s. sequences

in D [11]. He showed that if $\{a_n\}_{n=1}^\infty$ is a nondecreasing sequence of points in the interval $I = (0, 1)$ such that $\Sigma_{n=1}^\infty (1 - a_n) < \infty$ then one could construct a u.s. sequence, $\{\lambda_n\}_{n=1}^\infty$, with $|\lambda_n| = a_n$. Using a somewhat simpler argument, we obtain a similar, slightly weaker, result for α-separated sequences.

THEOREM 6.4. *Suppose* $\{a_n\}_{n=1}^\infty$ *is a nondecreasing sequence of points in* $I = (0, 1)$ *and* $\Sigma_{n=1}^\infty (1 - a_n)^p < \infty$ *for some* $p < 1$, *then there exists an* α-*separated sequence* $\{\lambda_n\}_{n=1}^\infty \subset D$ *such that* $|\lambda_n| = a_n$.

PROOF. We may as well assume $p > \frac{1}{2}$. For $k = 1, 2, 3, \cdots$ let

$$I_k = \{x \in I: 1 - 1/2^{k-1} < x \leqslant 1 - 1/2^k\}.$$

By throwing out a finite number of points of $\{a_n\}_{n=1}^\infty$, we can assume the sequence $\{a_n\}_{n=1}^\infty$ is embedded in another nondecreasing sequence of points in I, $\{\beta_n\}_{n=1}^\infty$, where $\{\beta_n\}_{n=1}^\infty$ has the following properties:
  (1) $\beta_n \geqslant 1 - 1/2^N$ for some fixed N,
  (2) $(1/2^k)^{1-p} \leqslant \Sigma_{\beta_i \in I_k}(1 - \beta_i)^p$ for $k = N, N + 1, \cdots$,
  (3) $\Sigma_{i=1}^\infty (1 - \beta_i)^p \leqslant \frac{1}{2}$.
(To see how to do this, take N such that $\Sigma_{n=N}^\infty (1/2^k)^{1-p} < 1/8$, and throw out a finite number of points of $\{a_n\}$ so that $a_n \geqslant 1 - (\frac{1}{2})^N$ for all n and $\Sigma_n (1 - a_n)^p \leqslant \frac{1}{4}$. Since $p \geqslant \frac{1}{2}$, $(1/2^k)^p \leqslant (1/2^k)^{1-p}$; so for each k, $k = N, N + 1, \cdots$, there exists an integer r_k such that

$$(1/2^k)^{1-p} \leqslant r_k (1/2^k)^p \leqslant 2(1/2^k)^{1-p}.$$

Construct the sequence

$$\{\gamma_j\}_{j=1}^\infty = \Big\{ \underbrace{1 - \frac{1}{2^N}, \cdots, 1 - \frac{1}{2^N}}_{r_N \text{ times}}, \cdots, \underbrace{1 - \frac{1}{2^k}, \cdots, 1 - \frac{1}{2^k}}_{r_k \text{ times}},$$
$$\underbrace{1 - \frac{1}{2^{k+1}}, \cdots, 1 - \frac{1}{2^{k+1}}}_{r_{k+1} \text{ times}}, \cdots \Big\};$$

then

$$\left(\frac{1}{2^k}\right)^{1-p} \leqslant \sum_{\gamma_j \in I_k} (1 - \gamma_j)^p = r_k \left(\frac{1}{2^k}\right)^p \leqslant 2\left(\frac{1}{2^k}\right)^{1-p}.$$

Let $\{\beta_n\}_{n=1}^\infty$ be the sequence obtained from the points of $\{\gamma_j\}_{j=1}^\infty \bigcup \{\alpha_i\}_{i=1}^\infty$ reordered so as to form a nondecreasing sequence. Then

$$\sum_{n=1}^{\infty} (1 - \beta_n)^p = \sum_{n=1}^{\infty} (1 - a_n)^p + \sum_{n=1}^{\infty} (1 - \gamma_n)^p$$

$$\leqslant \frac{1}{4} + 2 \sum_{k=N}^{\infty} \left(\frac{1}{2^k}\right)^p \leqslant \frac{1}{2}.$$

The other two properties of $\{\beta_n\}_{n=1}^{\infty}$ are easily verified.)

We construct a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset D$ such that $\{\lambda_n\}_{n=1}^{\infty}$ is α-separated and $|\lambda_n| = \beta_n$ for $n = 1, 2, \cdots$.

Let

$$\sigma_1 = \sum_{i=1}^{\infty} (1 - \beta_i)^p, \quad \sigma_2 = \sum_{i=2}^{\infty} (1 - \beta_i)^p, \quad \cdots, \quad \sigma_n = \sum_{i=n}^{\infty} (1 - \beta_i)^p, \cdots$$

and let $\lambda_j \in \{z \in D: |z| = \beta_j\} \cap \{z \in \Delta: |1 - z| = \sigma_j\}$. (Note that $\sigma_j \geqslant (1 - \beta_j)^p > 1 - \beta_j$ so the above intersection is nonempty.)

If $k > m$,

$$|\lambda_k - \lambda_m| \geqslant |1 - \lambda_k| - |1 - \lambda_m| = \sigma_k - \sigma_m = \sum_{n=m}^{k-1} (1 - \beta_n)^p.$$

Similarly, if $k < m$, then $|\lambda_k - \lambda_m| \geqslant \sum_{n=k}^{m-1} (1 - \beta_n)^p$.

We define $\alpha_n \in H^{\infty}(D)$ by

$$\alpha_n(z) = \frac{\bar{\lambda}_n}{|\lambda_n|} \frac{\lambda_n - z}{1 - \bar{\lambda}_n z};$$

then $|\alpha_n(z)| \leqslant 1$ for all z, $\alpha_n(\lambda_n) = 0$, and since $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$ the usual arguments show $\sum_{n=1}^{\infty}(1 - \alpha_n(z))$ converges uniformly on compact subsets of D.

We are interested in $\sum_{n=1}^{\infty} |1 - \alpha_n(\lambda_k)|$. If $n \neq k$,

$$|1 - \alpha_n(\lambda_k)| = \frac{1 - |\lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} | |\lambda_n| + \bar{\lambda}_n \lambda_k|$$

$$\leqslant 2 \frac{(1 - |\lambda_k|)}{|1 - \bar{\lambda}_n \lambda_k|} = 2 \frac{(1 - \beta_n)}{|\lambda_n - \lambda_k|} \left| \frac{\lambda_n - \lambda_k}{1 - \bar{\lambda}_n \lambda_k} \right|$$

$$\leqslant 2 \frac{1 - \beta_n}{|\lambda_n - \lambda_k|}.$$

Now fix k. We consider three separate possibilities.

Case 1. $n > k$, then

$$\frac{1}{2}\,|1 - \alpha_n(\lambda_k)| \leqslant \frac{1 - \beta_n}{\sum_{j=k}^{n-1}(1 - \beta_j)^p} \leqslant \frac{1 - \beta_n}{(n-k)(1 - \beta_{n-1})^p}$$

$$\leqslant \frac{1 - \beta_n}{(n-k)(1 - \beta_n)^p} = \frac{(1 - \beta_n)^{1-p}}{n-k}$$

$$\leqslant \frac{(1 - \beta_{n-k})^{1-p}}{n-k}\,.$$

So

$$\sum_{n=k+1}^{\infty} |1 - \alpha_n(\lambda_k)| \leqslant 2 \sum_{j=1}^{\infty} \frac{(1 - \beta_j)^{1-p}}{j}$$

but $(1 - \beta_j)^{1-p}$ is $p/(1 - p)$ summable $(p > \frac{1}{2}$ implies $p/(1 - p) > 1)$ and $1/j$ is $q = p/(2p - 1)$ summable; so by Hölder's inequality,

$$\sum_{n=k+1}^{\infty} |1 - \alpha_n(\lambda_k)| \leqslant 2 \sum_{j=1}^{\infty} \frac{(1 - \beta_j)^{1-p}}{j}$$

$$\leqslant 2\left(\sum_{j=1}^{\infty}(1 - \beta_j)^p\right)^{(1-p)/p}\left(\sum_{j=1}^{\infty}\left(\frac{1}{j}\right)^q\right)^{1/q} = C_1.$$

Cases 2 and 3. $n < k$, assume $|\lambda_k| = \beta_k \in I_j$, i.e. $1/2^j \leqslant (1 - \beta_k) \leqslant 1/2^{j-1}$.

Case 2. $1 - |\lambda_n| = (1 - \beta_n) \leqslant 1/2^{j-2}$. Then $(1 - \beta_n) \leqslant 4(1 - \beta_k)$, and, therefore

$$\frac{1}{2}\,|1 - \alpha_n(\lambda_k)| \leqslant \frac{1 - |\lambda_n|}{|\lambda_n - \lambda_k|} \leqslant \frac{1 - \beta_n}{\sum_{i=n}^{k-1}(1 - \beta_i)^p} \leqslant \frac{4(1 - \beta_k)}{(k-n)(1 - \beta_{k-1})}$$

$$\leqslant \frac{4(1 - \beta_k)}{(k-n)(1 - \beta_k)^p} = \frac{4(1 - \beta_k)^{1-p}}{k-n} \leqslant \frac{4(1 - \beta_{k-n})^{1-p}}{k-n}\,.$$

So

$$\sum_{\{n:\beta_n \in I_{j-1} \cup I_j;\, n<k\}} |1 - \alpha_n(\lambda_k)| \leqslant 8 \sum_{i=1}^{\infty} \frac{(1 - \beta_i)^{1-p}}{i} \leqslant 8C_1.$$

Case 3. $|\lambda_n| = \beta_n \in I_N \cup \cdots \cup I_{j-2}$, i.e. $(1 - \beta_n) > 1/2^{j-2}$.

$$\sum_{\{n:\beta_n \in I_N \cup \ldots \cup I_{j-2}\}} |1 - \alpha_n(\lambda_k)|$$

$$= \sum_{\{n:\beta_n \in I_N\}} + \sum_{\{n:\beta_n \in I_{N+1}\}} + \cdots + \sum_{\{n:\beta_n \in I_{j-2}\}}.$$

If $\beta_n \in I_i$,

$$\frac{1}{2} |1 - \alpha_n(\lambda_k)| \leqslant \frac{1 - \beta_n}{\sum_{l=n}^{k-1}(1 - \beta_l)^p} \leqslant \frac{1 - \beta_n}{Q}$$

where $Q = \sum_{\beta_l \in I_{i+1}}(1 - \beta_l)^p \geqslant (1/2^{i+1})^{1-p}$ (by the second property of the sequence $\{\beta_l\}_{l=1}^\infty$).

But $\beta_n \in I_i$ implies $(1 - \beta_n) \leqslant 1/2^{i-1}$. So

$$\frac{1}{2} |1 - \alpha_n(\lambda_k)| \leqslant \frac{(1 - \beta_n)^{1-p}}{(1/2^{i+1})^{1-p}} (1 - \beta_n)^p$$

$$\leqslant \frac{(1/2^{i-1})^{1-p}}{(1/2^{i+1})^{1-p}} (1 - \beta_n)^p \leqslant 4(1 - \beta_n)^p.$$

Hence

$$\sum_{\beta_n \in I_N \cup \ldots \cup I_{j-2}} |1 - \alpha_n(\lambda_k)| \leqslant 8 \sum_{r=1}^\infty (1 - \beta_n)^p \leqslant 4.$$

Consequently $\sum_{n=1}^\infty |1 - \alpha_n(\lambda_k)| \leqslant 4 + 8C_1 + 1 + C_1 = 9C_1 + 5$.

The proof is complete except to point out that in case (1) we showed that if $n > k$, $|1 - \alpha_n(\lambda_k)| \leqslant 1 - \beta_n \leqslant \frac{1}{2}$ hence $|\alpha_n(\lambda_k)| \geqslant \frac{1}{2}$. Clearly, if $n < k$, $|\alpha_n(\lambda_k)| = |\alpha_k(\lambda_n)| \geqslant \frac{1}{2}$, so property (2) of Definition 2.4 is satisfied.

REMARK. An application of Theorem 6.4 shows that general interpolating sequences in D^n ($n \geqslant 2$) need not be α-separated. Recall that in D, a sequence $S = \{a_i\}_{i=1}^\infty$ is u.s. iff

$$\operatorname*{Inf}_i \prod_{i=1; j \neq i}^\infty \left| \frac{a_i - a_j}{1 - a_i \bar{a}_j} \right| = \operatorname*{Inf}_i \prod_{j=1; j \neq i}^\infty \rho(a_i, a_j) > 0.$$

Since the metric, ρ, is defined in D^n, we can define a sequence $S = \{a_i\}_{i=1}^\infty$ in D^n to be "product uniformly separated" (p.u.s.) if $\operatorname{Inf}_i \Pi_{j=1; j \neq i}^\infty \rho(a_i, a_j) > 0$. If $S \subset D^n$ is α-separated (with $\alpha_1, \alpha_2, \alpha_3, \cdots$ the functions of Definition 2.4), then $\rho(a_i, a_j) > |\alpha_i(a_i)|$, and the remarks following Definition 2.4 guarantee $\operatorname{Inf}_i \Pi_{j=1; j \neq i}^\infty \rho(a_i, a_j) > 0$, so S is p.u.s.

Since p.u.s. is equivalent to u.s. and hence interpolation in D, one might ask whether u.s. sequences in D^n are p.u.s. and whether p.u.s. sequences in D^n are interpolating (w.r.t. C). We use Theorem 6.4 to give an example of a general interpolating (hence u.s.) sequence in D^2 which is not p.u.s. Since the sequence is not p.u.s. it is not α-separated, so α-separation is sufficient but not necessary for general interpolation in D^n for $n \geqslant 2$.

Let $S_1 = \{\lambda_n\} \subset D$ be the α-separated sequence constructed by Theorem 6.4 with the property that $|\lambda_n| = 1 - 1/n^2$. Then S_1 is a general interpolating

sequence and by Proposition 4.1, $S = S_1 \times S_1 \subset D^2$ is also a general interpolating sequence (and consequently interpolating and u.s.). We show that S is not p.u.s.

Noting that $\lambda_1 = 0$, consider

$$\sum_{i,j=1}^{N} (1 - \rho_2(\langle 0, 0 \rangle, \langle \lambda_i, \lambda_j \rangle)) \geqslant \sum_{i=2}^{N} \sum_{j=1}^{i} (1 - \rho_2(\langle 0, 0 \rangle, \langle \lambda_i, \lambda_j \rangle))$$

$$= \sum_{i=2}^{N} \sum_{j=1}^{i} (1 - \rho_1(0, \lambda_i)) = \sum_{i=2}^{N} \sum_{j=1}^{i} (1 - |\lambda_i|)$$

$$= \sum_{i=2}^{N} \frac{i}{i^2} = \sum_{i=2}^{N} \frac{1}{i}.$$

Consequently, the sum $\Sigma_{\langle \lambda_i, \lambda_j \rangle \in S}(1 - \rho_2(\langle 0, 0 \rangle, \langle \lambda_i, \lambda_j \rangle))$ diverges, hence

$$\prod_{\lambda = \langle \lambda_i, \lambda_j \rangle \in S; \lambda \neq 0} \rho_2(\lambda, 0) = 0,$$

and S is not p.u.s.

7. **Other contexts.** In this section we briefly outline applications of the above results to contexts other than that of bounded analytic functions on polydisks. First we simply replace D^n with a multiply connected region of $\mathbf{C}$, then we replace the uniform algebras $H^\infty(D^n)$ by an arbitrary uniform algebra A.

Let Ω be an n-connected region in $\mathbf{C}$, bordered by n simple closed curves $C_1, C_2, \cdots, C_n$. Let $\Omega_1, \Omega_2, \cdots, \Omega_n$ be the complements in the extended plane of the closures of the components of the complement of Ω with $\partial \Omega_k = C_k$. Denote by $H^\infty(\Omega), H^\infty(\Omega, A), H^\infty(\Omega_k)$, and $H^\infty(\Omega_k, A)$ the algebras of bounded analytic $\mathbf{C}$ and A valued functions defined on Ω and Ω_k. We can define interpolating, general interpolating and uniformly separated sequences in Ω in a natural manner.

Let $\phi_k : \Omega_k \longrightarrow D$ conformally with $\phi_k(z) \longrightarrow T^1$ as $z \longrightarrow C_k$ (recall $T^1 = \{z \in \mathbf{C} : |z| = 1\}$) and define $\phi: \Omega \longrightarrow D^n$ by $\phi(z) = \langle \phi_1(z), \phi_2(z), \cdots, \phi_n(z) \rangle$.

DEFINITION 7.1. We say $W \subset \Omega$ is a wedge in Ω if $\phi(W)$ is contained in a wedge in D^n.

THEOREM 7.2. *Suppose* $S = \{a_i\}_{i=1}^{\infty} \subset \Omega$ *is u.s., then*
(a) *S is an interpolating sequence,*
(b) *if S is contained in a wedge in Ω, S is a general interpolating sequence.*

PROOF. First we show that $\phi(S)$ is u.s. in D^n.
It is well known that there exists a constant P depending only on Ω such

that $f \in H^{\infty}(\Omega)$ and $\|f\| \leqslant 1$ implies the existence of $f_1, f_2, \cdots, f_n; f_k \in H^{\infty}(\Omega_k)$ and $\|f_k\| \leqslant P$ such that for $z \in \Omega$, $f(z) = \Sigma_{k=1}^n f_k(z)$.

If $f_j \in H^{\infty}(\Omega)$ and $\|f_j\| \leqslant 1$ and $f_j(a_j) = 1/M$ and $f_j(a_i) = 0$ if $i \neq j$, then $f_j = \Sigma_{k=1}^n f_{j_k}$, $f_{j_k} \in H^{\infty}(\Omega_k)$, $\|f_{j_k}\| \leqslant P$. Define $F_j \in H^{\infty}(D^n)$ by

$$F_j(\langle z^{(1)}, \cdots, z^{(n)} \rangle) = \sum_{k=1}^n f_{j_k}(\phi_k^{-1}(z^{(k)})).$$

Then $\|F_j\| \leqslant nP$ and

$$F_j(\phi(a_i)) = \sum_{k=1}^n f_{j_k}(\phi_k^{-1}(\phi_k(a_i)))$$

$$= \sum_{k=1}^n f_{j_k}(a_i) = f_j(a_i) = \begin{cases} 1/M & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases}$$

so $\phi(S)$ is u.s. in D^n.

For case (b) this is sufficient, since $\phi(S)$ is uniformly separated and contained in a wedge, and Theorem 5.9 can be applied to show $\phi(S)$ is a general interpolating sequence in D^n. Since ϕ is an analytic map from Ω to D^n, S will be a general interpolating sequence in Ω.

For case (a) we note that $|\phi_k(z)| \longrightarrow 1$ for some k implies $|\phi_j(z)|$ will be bounded away from 1 for $j \neq k$. Therefore, if S_j is a sequence in Ω with all its cluster points on C_j, and S_j' is another sequence in Ω with all its cluster points on $C_1 \cup C_2 \cup \cdots \cup C_{j-1} \cup C_{j+1} \cup \cdots \cup C_n$, then $\mathrm{Cl}(S_j) \cap \mathrm{Cl}(S_j') = \emptyset$. Consequently, by using Corollary 4.4, we can assume S has all its cluster points on C_1. In that case, there will exist ϵ, such that $|\phi_j(S)| < 1 - \epsilon$ for $j = 2, 3, \cdots, n$. So, if $0 < \tau < \epsilon/2$, $\phi(S) \subset D \times W_\tau \times W_\tau \times \cdots \times W_\tau$. Where $W_\tau = \{z \in D: (1 - |z|^2)/|1 - z^2| \geqslant \tau\}$. Thus, $\phi(S)$ is contained in a near-wedge in D^n, and by Theorem 5.9, is an interpolating sequence. As above, this implies S is an interpolating sequence in Ω.

DEFINITION 7.3. If $z, w \in \Omega$ define the pseudo-hyperbolic distance, ρ_Ω, between z and w to be $\rho_\Omega(z, w) = \mathrm{Sup}\ \{|f(z)|: f \in H^{\infty}(\Omega), \|f\| \leqslant 1, f(w) = 0\}$.

COROLLARY 7.4. *Suppose* $S = \{a_i\}_{i=1}^{\infty}$ *is a sequence in* Ω *with the property that* $\mathrm{Inf}_{i \neq j} \rho_\Omega(a_i, a_j) > 0$. *If* S *is contained in a finite union of wedges in* Ω *then* S *is a general interpolating sequence.*

PROOF. We note that there is a constant M, depending only on Ω, such that for any $f \in H^{\infty}(\Omega)$, where $\|f\| \leqslant 1$, there exists $F \in H^{\infty}(D^n)$ such that $\|F\| \leqslant M$ and $F \circ \phi = f$. (The proof of this fact is exactly the same as the argument of Theorem 7.2 that the image under ϕ of a u.s. sequence in Ω is u.s. in D^n; $M = nP$.) Consequently, for any z, w, $\rho_n(\phi(z), \phi(w)) \geqslant (1/M)\rho_\Omega(z, w)$.

Therefore, $\phi(S)$ satisfies the hypothesis of 5.16 and is a general interpolating sequence, from which it follows that S is a general interpolating sequence in Ω.

As a final generalization of the results of the first six sections, we note that $H^\infty(D^n)$ is just a uniform algebra and D^n is a subset of its maximal ideal space, m_n.

DEFINITION 7.5. If A is a uniform algebra with maximal ideal space m_A, a discrete sequence, $S = \{a_i\}_{i=1}^\infty \subset m_A$ is an interpolating sequence for A if the map $T: A \to l^\infty$ given by $T(X) = \{X(a_i)\}_{i=1}^\infty$ is surjective. If S is an interpolating sequence for A, we again will have a constant, M, with the property that if $\{\lambda_i\}_{i=1}^\infty \in l^\infty$ and $\mathrm{Sup}_i|\lambda_i| \leqslant 1$, then there will be $X \in A$ such that $\|X\| \leqslant M$ and $X(a_i) = \lambda_i$. M will be called an interpolating bound for S.

We note that the functions in A are all $\mathbf{C}$ valued, so we will be interested only in "simple" interpolating sequences.

DEFINITION 7.6. A discrete sequence $S = \{a_i\}_{i=1}^\infty \subset m_A$ is uniformly separated for A if there exists a constant M and elements, $X_1, X_2, \cdots \in A$, such that for all i, $\|X_i\| \leqslant M$, $X_i(a_i) = 1$, and $X_i(a_j) = 0$ for $j \neq i$. M is called a u.s. bound for S.

We noted that for $A = H^\infty(D^n)$ and $S \subset D^n$, we had no example of a u.s. sequence which is not interpolating. However, if $A(D)$ is the algebra of continuous functions on the closed unit disk, $\overline{D}$, which are analytic inside D, and S is a sequence contained in D which is u.s. for $H^\infty(D)$ and has a finite number of limit points in $\overline{D}$, then S is u.s. for the algebra $A(D)$, but it is not an interpolating sequence for $A(D)$. The problem is that unlike in $H^\infty(D)$, there are no functions in A which "adhere" to bounded sequences of functions. This difference suggests a minimal necessary condition that a sequence be interpolating:

PROPOSITION 7.7. *A necessary condition that a sequence $S \subset m_A$ be interpolating is the following weak adherence condition:*

(W)
> *There exists a constant M such that for any sequence of elements $\{X_i\}_{i=1}^\infty$ in the unit ball of A, there exists $X \in A$ with $\|X\| \leqslant M$ such that a subsequence of the $\{X_i\}_{i=1}^\infty$ converges pointwise on S to X.*

PROOF. The proof follows from the definition of interpolating sequence and the following lemma.

LEMMA 7.8. *For $\{\lambda^{(k)}\}_{k=1}^\infty$ a bounded sequence of elements of l^∞, there exists a subsequence which converges "pointwise" to an element in l^∞; i.e. if $\lambda^{(k)} = \{\lambda_i^{(k)}\}_{i=1}^\infty$ then there exists an indexing set $k_1, k_2, \cdots$ and a sequence $\lambda = \{\lambda_i\}_{i=1}^\infty \in l^\infty$ such that $\lambda_i^{(k_j)} \underset{j}{\to} \lambda_i$ for each i.*

PROOF. The proof follows from a standard diagonalization argument.

It is well known that the algebras $H^\infty(D)$ and $H^\infty(D^n)$ are sequentially compact with respect to pointwise convergence in D (or D^n). This is, of course, a much stronger statement than assertion (W), indeed it implies that (W) holds for any subset of D (or D^n) and in all cases $M = 1$. On the other hand, as we have seen, condition (W) is not satisfied by sequences in D for the algebra $A(D)$.

Lemma 3.2, Corollary 3.4, Theorem 4.3, Corollary 4.4, and Proposition 4.5 and their proofs all have direct and natural analogues for interpolating sequences in m_A. We do not know of any neat hypothesis (like that of Theorem 5.9, for example) under which u.s. implies interpolation. However, we do have a generalization of Theorem 2.5.

DEFINITION 7.9. If A is a uniform algebra and $S = \{a_n\}_{n=1}^\infty \subset m_A$, then S is α separated if there exist constants M and $p > 0$ and a sequence $\{\alpha_n\}_{n=1}^\infty \subset A$ such that

(1) $\|\alpha_n\| \leqslant 1$ for all n.

(2) $\alpha_n(a_n) = 0$ but $|\alpha_n(a_m)| > p$ if $n \neq m$.

(3) $\Sigma_{n=1}^\infty |1 - \alpha_n(a_m)| \leqslant M$ for all m.

(4) Condition (W) holds for S.

THEOREM 7.10. *If A is a uniform algebra and S is an α-separated sequence in m_A, then S is an interpolating sequence for A.*

PROOF. S α-separated implies condition (W) holds, so let M_W be the constant associated with condition (W) for S. We first show that S is u.s. For fixed j, consider

$$x_j^N = \prod_{n=1;\, n \neq j}^N \alpha_n$$

$x_j^N \in A$, $\|x_j^N\| \leqslant 1$ and $x_j^N(a_n) = 0$ if $n \neq j$ and $1 \leqslant n \leqslant N$, while (3) of Definition 7.8 implies $|x_j^N(a_j)| > e^{-M/p}$. Applying condition (W) to $\{x_j^N\}_{N=j+1}^\infty$ we get $x_j \in A$, $\|x_j\| \leqslant M_W$, such that $x_j(a_n) = 0$ if $n \neq j, n = 1, 2, \cdots$ and $|x_j(a_j)| > e^{-M/p}$. So S is u.s. with bound $1/\delta = M_W e^{M/p}$. Therefore we can define $f_1, f_2, \cdots \in A$ such that $\|f_j\| \leqslant 1$ and $f_j(a_i) = 0$ if $i \neq j$, and $f_i(a_i) = 0$.

The rest of the proof proceeds in a manner similar to the proof of Theorem 2.5 (see Theorem 2.2 of [10]) except that we have no guarantee that the infinite products involved will converge. Hence we must show that interpolations can take place on finite subsequences and that all these interpolations can be performed with functions of a given norm. An application of condition (W) then yields the final results.

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